In this paper, we explore various methods of measuring noise power in RF systems. We discuss some of the systems commonly used to make such measurements, then analyze each system’s architecture to obtain the measurement bias and uncertainty for six different combinations of noise models (real and complex noise) and methods of averaging (linear power, magnitude, and logarithmic). Equipped with this knowledge, users can correct for post-measurement biases and determine the number of averages required to yield a desired uncertainty level.

Introduction
The impact of noise on RF systems is a universal issue. Whether in communications, radar, telemetry, or remote sensing, noise has the potential to disrupt our measurements and cause uncertainty in the resultant data. In order to understand the impact of noise, it is often desirable to measure the noise power in the environment or in an RF system. It is common to characterize RF components and systems by their noise figure (when an antenna is not involved) or gain over temperature (G/T) (when an antenna is involved) using methods such as the cold source method (also known as the direct noise method) and the Y-factor method [1]–[4]. These methods require one or more measurements of noise power at the output of a device under test.

Noise power measurements in RF systems are sometimes conducted without a thorough understanding of the bias and uncertainty associated with such measurements. Many books and journal articles have analyzed the uncertainties associated with noise figure, noise temperature, and noise power measurements [3], [5]–[7]. All of these measures rely on some form of non-coherent averaging to estimate noise power. Some of the methodologies perform this averaging at multiple stages.

In modern instrumentation, such as modern spectrum analyzers, there are different options on how the instrument performs this averaging. Each method has a different impact on the bias and uncertainty of the reported noise power, depending on the instrument’s architecture. The goal of this paper is to analyze and summarize these biases and uncertainties, which will then enable the user to conduct such tests with greater confidence. When properly understood, the biases can be removed, and the uncertainty bounds can give the user an understanding of the accuracy of their noise power measurement.

This paper begins with a brief discussion of noise theory. We then look at different methods of directly measuring RF noise power and the instruments associated with these methods. We derive equations used to compute the biases and uncertainties associated with each of these measurement methods, and we present Monte Carlo simulation results that support the results of the derivations.

Noise Theory
For many electrical devices, noise at RF frequencies (e.g., 3 kHz to 300 GHz) is typically dominated by thermal noise, sometimes called Johnson-Nyquist noise [1], [8]. The voltage representation of this noise can be modeled as a zero-mean stationary Gaussian process, often denoted as [9]:

\[ X(t) \sim \mathcal{N}(0, \sigma^2_x). \] (1)

The variance, \( \sigma^2_x \), is equal to the noise power, given by [10]

\[ P_x = \sigma^2_x = E\left[|X(t)|^2\right] = kTB, \] (2)

where \( k \) is Boltzmann’s constant, \( T \) is the effective noise temperature of the system, \( B \) is the equivalent noise bandwidth, and \( E[\cdot] \) indicates the expected value operator.

In some systems and frequencies, flicker noise dominates, but this noise may also be modeled as Gaussian [11]. Thus, the Gaussian model will be applied to all RF noise, and the variance of that Gaussian process is what we seek to measure in an RF noise power measurement.

If this noise process is sent through a Hilbert transform and recombined to yield the complex envelope of an ideal in-phase/quadrature (I/Q) mixdown, whether implemented in analog or digital circuitry, the result will be [12]:

\[ X_c(t) = \frac{1}{\sqrt{2}} \left[ X(t) + jX(t) \right] e^{j\omega t}, \] (3)
where $\hat{X}_r(t)$ is the Hilbert transform of $X_r(t)$ and $\omega_c$ is the frequency shift, typically given by the carrier frequency. In the case of a noise process, there is no carrier, but we are often concerned with the noise power centered around some frequency $\omega_c$.

The noise bandwidth is likely set by a filter stage within the RF receive chain. This means that the resultant noise is colored. However, if we assume that the noise samples are acquired at a rate that is no greater than the inverse of the filter’s two-sided equivalent noise bandwidth, then we can assume that the noise seen by the receiver is approximately white [13]. This is the case for a digital receiver that is critically sampled. Therefore, for the subsequent analyses, we assume in all cases that we are dealing with white Gaussian noise (WGN) and not colored or oversampled noise.

The formation of $X_c(t)$ could be implemented in either analog or digital circuitry, but is commonly done digitally and may be modeled as an I/Q mix-down followed by a digital low-pass filter and a decimator [12]. The term $X_c(t)$ is a circularly symmetric complex Gaussian process, i.e., $X_c(t) \sim CN(0, \sigma^2_c)$. Since we have assumed the noise spectrum to be white, this random process can be considered complex white Gaussian noise (CWGN). Although the total noise power does not change, our new complex Gaussian process is now composed of two independent real Gaussian processes given by:

$$I(t) = \text{Re}\{X_c(t)\} \sim N\left(0, \frac{\sigma^2_c}{2}\right)$$

$$Q(t) = \text{Im}\{X_c(t)\} \sim N\left(0, \frac{\sigma^2_c}{2}\right)$$

where each real Gaussian process $I(t)$ and $Q(t)$ has half the noise power of the original real noise process $X_r(t)$ [12].

Because $X_c(t)$ is a stationary process, we can drop the dependence on time and use the random variables $X_r$ and $X_c(t)$ interchangeably to represent the real white Gaussian noise process. In the same fashion, we can use the random variables $X_r$ and $X_c(t)$ interchangeably to represent the complex white Gaussian noise process.

**Methods of Measuring RF Noise Power**

Various devices can be used to measure RF noise power. The following sub-sections summarize a few of these devices and provide a block diagram model for each device. We will use these models later to analyze the measured noise power bias and uncertainty for each device.

**Power Sensor**

A power sensor attached to a power meter is a common tool used to measure RF power. We assume that the power sensor is preceded by a bandpass filter with the same bandwidth as the video bandwidth of the power sensor to ensure that the sampled noise can be modeled as white noise. In certain cases, as described in [14], a Low Noise Amplifier (LNA) may be placed between the Device-Under-Test (DUT) and the power sensor to improve the sensitivity of the measurement. The behavior of a power sensor is similar to the behavior of a true-RMS voltmeter: the former reports power, whereas the latter reports root-mean-square voltage. An in-depth discussion on the theory of operation of diode-based, thermocouple-based, and thermistor-based power sensors can be found in [15].

Fig. 1 shows a block diagram for the power sensor and power meter setup. When queried to provide an individual power sample, the power sensor and power meter will report a single reading of the magnitude-squared value of the real noise process $X_r$.

Modern power meters are able to average successive power sensor readings in the linear power domain. Since the linear power value is proportional to the magnitude-squared value of the noise via a calibration constant, the method of averaging employed by this first device is an example of “real, magnitude-squared” averaging. As will be shown in the section entitled “Calculation of the Noise Power Biases and Variances,” the “real, magnitude-squared” method provides an estimate of the noise power with no bias.

**Full-wave Rectifier**

Another device that could be used to measure RF noise power, albeit a bit more theoretical than practical, is a device that...
primarily consists of a full-wave rectifier followed by a high-speed digitizer, as shown in Fig. 2. In this case, the device does not provide an estimate of the noise power directly; rather, this device reports the magnitude of $X_t$ at a given instant in time. If the device only allows averaging to be performed on the magnitude of $X_t$, then this is an example of “real, magnitude” averaging. Averaging the magnitude of the time domain signal and squaring the result will cause a bias in the estimate of the expected value of $|X_t|^2$, as will be shown later. If the path that includes the log amp is chosen, and averaging is performed on the resultant value, this is an example of “real, logarithmic” averaging, and a different bias to the estimated noise power will be observed.

**Swept-tuned Superheterodyne Spectrum Analyzer**

A third device that is commonly used to measure RF noise power is the swept-tuned superheterodyne spectrum analyzer. A simplified block diagram for this device is shown in Fig. 3 [16]. The process followed by the model can be summarized as follows:

- The real noise signal $X_r(t)$ is mixed with the LO frequency $\omega_0$, which forms $X_r(t)\cos(\omega_0 t)$.
- The resultant signal is passed through a variable bandwidth IF filter.
- The signal is then either directed to a logarithmic amplifier or passed directly to the envelope detector.
- Finally, the output of the envelope detector is directed either to a digitizer or to the display circuitry, depending on the architecture of the spectrum analyzer [16].

The output of the envelope detector is the envelope function, which is modeled as the absolute value of the pre-envelope function [17]:

$$X_{en}(t) = |Z(t)| = \sqrt{Z(t)Z^*(t)}. \quad (6)$$

The pre-envelope function $Z(t)$ is formed from the real signal and the Hilbert transform of the real signal [17]:

$$Z(t) = X_r(t)\cos(\omega_0 t) + jH\{X_r(t)\cos(\omega_0 t)\}, \quad (7)$$

where $H\{\cdot\}$ represents the Hilbert transform. Thus, the output of the envelope detector when the linear path is selected can be expressed as:

$$X_{en}(t) = \sqrt{|X_r(t)\cos(\omega_0 t)|^2 + |H\{X_r(t)\cos(\omega_0 t)\}|^2} \approx |X_r(t)|. \quad (8)$$

The output of the envelope detector is therefore distributed as $|X_r(t)| - |X_r|$, since we have assumed that $|X_r(t)|$ is a stationary process with an autocorrelation function equal to zero for all time lags.

Similar to the rectifier-plus-digitizer model, to estimate the noise power, the swept-tuned superheterodyne spectrum analyzer must be able to average successive measurements. Averaging successive samples of the magnitude of the complex signal is an example of “complex, magnitude” averaging. Averaging successive samples of $20\log_{10}|X_r|$, in the case of the log-amp path, is an example of “complex, logarithmic” averaging. Both “complex, magnitude” and “complex, logarithmic” averaging will introduce a bias in the reported noise power value. While many modern spectrum analyzers will perform the averaging on the power value [18] and thus avoid the bias, some older models allow the user to perform trace averaging on the equivalent of $|X_r|$ and thus do incur the bias in the reported noise power value [16].

**Real-Time Spectrum Analyzer**

The last device to be analyzed is the real-time spectrum analyzer [19]. Although there are various real-time spectrum analyzer architectures, for simplicity, we will focus on one specific architecture called the Fourier analyzer, since the block diagram of a Fourier analyzer is the easiest to construct. Other real-time spectrum analyzers, like the parallel-filter analyzer and the vector signal analyzer (VSA), only differ in the way that the signal is filtered or down-converted prior to digitization. Thus, once we understand how digitization and digital downconversion impact the distribution of the sampled noise signal, we can apply
this understanding to all real-time spectrum analyzer implementations. A simplified block diagram of a Fourier analyzer is shown in Fig. 4 [16].

The Fourier analyzer digitizes the signal immediately after the signal has passed through an anti-alias low-pass filter. The Fourier analyzer then processes the digital signal to create an analytic signal using the Hilbert transform [16]. This signal is then converted to baseband via a complex mix-down and passed through a digital low-pass filter to yield $X_c$.

The Fourier analyzer can average successive samples of $X_c$ to estimate the power of a signal. The Fourier analyzer performs this averaging in either software or firmware, which gives the instrument the ability to perform the averaging on $X_r$. This is an example of “complex, magnitude-squared” averaging, and, as we will show later, there is no bias in the estimated noise power for this scenario.

Summary of Noise Measurement Devices

Table 1 summarizes the devices discussed above. Note that each device implements one or more of six possible averaging scenarios. Each device may either average real or complex noise. Also, depending on the device, the method of averaging the noise signal may be “magnitude-squared,” “magnitude,” or “logarithmic” averaging.

Calculation of the Noise Power Biases and Variances

In this section, we analyze the six cases summarized in Table 1. The following sub-sections analyze the mean and variance of each of these cases. The means yield measurement biases, and the variances yield measurement uncertainties. To validate the derivations, we present Monte Carlo simulation results and compare them to the analytical expressions.

Magnitude-Squared Averaging of Real Noise

In the context of this paper, the term “real noise” is used to represent the signal $X_r$, which is assumed to be a white Gaussian stationary random process distributed as $\mathcal{N}(0, \sigma_r^2)$.

Previously, we described how modern power meters have the ability to average successive samples collected by the power sensor. That averaging is performed on $|X_r|^2$. Because $X_r$ is a Gaussian distribution, the distribution of $|X_r|^2$ is chi-squared with one degree of freedom, where the underlying normal distribution has a variance of $\sigma_r^2$. Using the properties of the chi-squared distribution found in [20], we can compute the expected value of $|X_r|^2$ as follows:

$$E[|X_r|^2] = \frac{\sigma_r^2}{2}$$

As shown in (2), the true noise power of the noise process is $P_n = \sigma_r^2$. Thus, the noise power reported by a “real, magnitude-squared” averaging device, given a sufficient amount of time, will approach the true noise power.

Engineers, scientists, and metrologists also seek to understand the uncertainty of any given measurement. To determine the uncertainty of the power sensor’s reported sample mean, we must compute the standard deviation of a single noise power measurement. Again, since $|X_r|^2$ is chi-squared with one degree of freedom, where the underlying normal distribution’s variance is $\sigma_r^2$, the variance of a single noise power measurement for a “real, magnitude-squared” device is $2\sigma_r^2$ [20], and thus the standard deviation is $\sqrt{2}\sigma_r$. If $N$ samples are collected, then the standard deviation of the reported value of the noise power (i.e., the standard error of the reported noise power) is given by [20]:

$$\sigma_{\text{real,magnitude-squared}}(N \text{ samples}) = \frac{\sqrt{2}\sigma_r}{\sqrt{N}}$$

Table 1 – Summary of noise measurement devices and their averaging modalities

<table>
<thead>
<tr>
<th>Method of Averaging</th>
<th>Real noise</th>
<th>Complex noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnitude-squared</td>
<td>Power sensor</td>
<td>Real-time spectrum analyzer</td>
</tr>
<tr>
<td>Magnitude</td>
<td>Full-wave rectifier, linear path</td>
<td>Super-het spectrum analyzer, linear path</td>
</tr>
<tr>
<td>Logarithmic</td>
<td>Full-wave rectifier, log path</td>
<td>Super-het spectrum analyzer, log path</td>
</tr>
</tbody>
</table>
This method achieves the Cramer-Rao Lower Bound (CRLB) for a standard deviation estimator of white Gaussian noise [13]. The error, or uncertainty, of this measurement in decibels (dB) is given by the difference between the worst-case estimate for a given probability and the true value in dB, i.e.,

$$U^* = 10 \log_{10} \left( \mu_x + k \sigma_{n} \right) - 10 \log_{10} \left( \mu_x \right),$$

where $k$ is the number of standard deviations to include in the uncertainty, $\mu_x$ is the mean value of the magnitude-squared samples, and $\sigma_{n}$ is the standard deviation of the sample mean. The positive and negative uncertainties can be written in combined form as:

$$U = 10 \log_{10} \left( \frac{\mu_x \pm k \sigma_{n}}{\mu_x} \right) = 10 \log_{10} \left( 1 \pm k \frac{\sigma_{n}}{\mu_x} \right).$$

Therefore, when averaging the output of a magnitude-squared device, we can write the uncertainty as:

$$U_{\text{real,magnitude-squared}} = 10 \log_{10} \left( 1 \pm k \frac{2}{\sqrt{N}} \sigma_x \right).$$

Note that the uncertainty of the measurement in dB is independent of the actual noise power we are trying to measure. This will be the case for all of the systems analyzed in this paper. The Taylor series first-order approximation:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \approx x,$$

becomes increasingly valid as $x \rightarrow 0$, which occurs as $N \rightarrow \infty$. Thus, we can write:

$$U_{\text{real,magnitude-squared}} \approx \pm 10 k \frac{2}{\sqrt{N}} \log_{10} e \approx \pm 0.69149 k \sqrt{N} \text{ dB.}$$

Using the above equation, we can compute the number of samples ($N$) required to achieve a desired level of uncertainty. This is valid for a white noise signal. If we have oversampled noise, we would want to multiply the computed value of $N$ by the oversample factor, given by dividing the sample rate by the two-sided equivalent noise bandwidth of the receiver.

**Magnitude Averaging of Real Noise**

If the noise is sampled with our conceptual device of a full-wave rectifier plus digitizer, then the result is $|X_n|$ when the signal takes the linear path. If this device were to report power by first taking the time-average of the magnitude of the signal, and then squaring that result, this would be a device that performs magnitude averaging on the real noise signal. If allowed to perform an infinite number of averages, this style of device would report the expected value of $|X_n|^2$.

To determine the bias incurred by employing “real, magnitude” averaging, we must look at the distribution of $|X_n|$. Since $X$ is a white Gaussian random process, the distribution of $|X_n|$ will be a half-normal (also known as a “folded”) distribution. The expected value of a half-normal distribution is [21]:

$$E[|X|] = \sigma \sqrt{2 / \pi}.$$

This is valid for a white noise signal. If we have oversampled samples, and then squaring that result, this would be a device that performs magnitude averaging on the real noise signal. Therefore, when averaging the output of a magnitude-squared device, we can write the uncertainty as:

$$U_{\text{real,magnitude}} = 10 \log_{10} \left( \sigma_x \frac{2}{\pi} \sigma_x \right) \approx -1.9612 \text{ dB.}$$
Based on (13), we can write the uncertainty in dB as:

$$U_{\text{real, magnitude}} = 20 \log_{10} \left( \sqrt{\frac{1 - \frac{2}{N} \sigma_x^2}{} \right.$$}
$$\left. + \frac{1}{\sqrt{N} \sigma_x^2} \right)$$
$$\approx \pm 20 k \sqrt{\frac{2}{N}} \log_{10} e = \pm 6.5623k \text{ dB}.$$ (24)

**Logarithmic Averaging of Real Noise**

If the noise is sampled with the full-wave rectifier plus digitizer along the log-amp path, then we have a case of “real, logarithmic” averaging. To derive the bias and variance of this case, we start with the probability density function (pdf) of real noise $X_r$, which has a mean value of zero and a variance of $\sigma_x^2$:

$$f_{x_r}(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{x^2}{2\sigma_x^2}}.$$ (25)

The random variable $X_r$ is converted to a dB representation, indicated by the random variable $Y_r$, by implementing the following equation:

$$Y_r = 20 \log_{10} \left( |X_r| \right).$$ (26)

Using this representation of $Y_r$, we may derive the mean and standard deviation of $Y_r$. To do so, it is not necessary to derive the probability density function of $Y_r$. Rather, if we can write $Y = f(X)$, then $E[Y] = E[f(X)]$, as described in [20].

The expected value of $Y_r$ is derived in Appendix A and results in the following expression:

$$E[Y_r] = 20 \log_{10} \left( \frac{1}{\sqrt{2\pi}\sigma_x} \right) e^{-\frac{1}{2\sigma_x^2}} dx$$
$$= 20 \log_{10} \sigma_x - \frac{10}{\ln(10)} \left[ y + \ln(2) \right].$$ (27)

where $\gamma = 0.5772157$ is the Euler-Mascheroni constant.

Comparing this result to the true noise power, we discover the noise power bias of a “real, logarithmic” averaging device is:

$$\text{BIAS}_{\text{real, magnitude}} [\text{dB}] = E[Y_r] - 10 \log_{10} (\sigma_x^2)$$
$$= 20 \log_{10} \sigma_x - \frac{10}{\ln(10)} \left[ y + \ln(2) \right] - 20 \log_{10} \sigma_x$$
$$= -5.5171 \text{ dB}.$$ (28)

To determine the uncertainty of a noise power value provided by a “real, logarithmic” averaging device, we must first compute the variance of a single measurement. The variance of $Y$ is calculated from the expected value and second moment of $Y$ as follows:

$$\sigma_y^2 = \text{Var}[Y] = E[Y^2] - E[Y]^2.$$ (29)

The second moment of $Y$ is derived in Appendix B and is given by:

$$E[Y^2] = \int_{-\infty}^{\infty} \left( 20 \log_{10} (|x|) \right)^2 \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{x^2}{2\sigma_x^2}} dx$$
$$= \frac{400}{\ln(10)} \left[ \ln^2 \left( \sqrt{2}\sigma_x \right) - \ln \left( \sqrt{2}\sigma_x \right)^2 \right] + \frac{1}{4} \left[ y + \ln(4) \right] + \frac{\ln^2 \left( \sqrt{2}\sigma_x \right) - \ln \left( \sqrt{2}\sigma_x \right)^2}{8}.$$ (30)

The expected value of $Y$ was previously computed and the result is shown in (27). The variance of $Y_r$, therefore, is given as:

$$\sigma_y^2 = E[Y_r^2] - E[Y_r]^2 = \frac{50\pi^2}{\ln(10)}.$$ (31)

and the standard deviation of $Y$, is:

$$\sigma_y = \frac{\sqrt{50\pi}}{\ln(10)} \approx 9.6476 \text{ dB}.$$ (32)

The reader will notice that the variance and standard deviation of the measured noise power in the case of a “real, logarithmic” device are independent of the noise power level. This is logical because we are analyzing the signal on a dB scale. An example of a sinusoidal amplitude-modulated signal will help illustrate this point. Let us assume a signal level of 1 V peak amplitude with a 10% modulation index. The signal’s amplitude will vary from 0.9 to 1.1 V, a total variation of 1.74 dB. If this signal is attenuated by 6 dB, then the peak signal amplitude will be reduced to 0.5 V amplitude, with an amplitude variation from 0.45 to 0.55 V. The total variation is still 1.74 dB, and will always be so, regardless of how much the amplitude-modulated signal is attenuated or amplified.

Similarly, the variation of a noise process when assessed on a logarithmic scale will always be the same regardless of the actual power level of the noise. This can be readily observed on a spectrum analyzer by examining the vertical spread of the noise floor. For a fixed number of averages, that spread is always the same regardless of the power level at which the spread is observed.

If $N$ samples of $Y$ are averaged, then the standard deviation of the reported value of the noise power can be computed as:

$$\sigma_{\text{real, magnitude}} [N \text{ samples}] = \frac{\sqrt{50\pi}}{\ln(10)\sqrt{N}} \approx 9.6476 \text{ dB}.$$ (33)

The standard deviation above is given in dB. Unlike the previous cases where we computed the spread of possible values converted to dB, we can write the k-r uncertainty directly as:

$$U_{\text{real, magnitude}} = \pm \frac{9.6476k}{\sqrt{N}} \text{ dB}.$$ (34)

**Magnitude-squared Averaging of Complex Noise**

Recall that $X$ denotes a complex random process. If this noise signal is sampled with an analog-to-digital converter (ADC)
and converted to successive samples of \( X \), as would be done when employing a real-time spectrum analyzer (Fig. 4) then the complex signal \( X \) can be written as:

\[
X = \text{Re}\{X\} + j\text{Im}\{X\} = I + jQ.
\]  

(35)

The variance of \( I \) and \( Q \) are identical, and are given as:

\[
\text{VAR}[I] = \text{VAR}[Q] = E[|I|^2] = E[|Q|^2] = \frac{\sigma^2}{2}.
\]  

(36)

Note that \(|I|^2\) and \(|Q|^2\) are distributed as chi-squared distributions with one degree of freedom, where each underlying normal distribution has a variance of \( \sigma^2/2 \).

Thus, the expected value of \( |X|^2 \) is given as:

\[
E[|X|^2] = E[|I|^2] + E[|Q|^2] = 2 \left( \frac{\sigma^2}{2} \right) = \sigma^2.
\]  

(37)

If the noise power is computed by taking the mean of the magnitude-squared of the complex signal, then the noise power would be reported as follows:

\[
P_{	ext{complex magnitude-squared}} = E[|X|^2] = \sigma^2.
\]  

(38)

This is an example of a “complex, magnitude-squared” method of averaging.

Comparing this result to the true noise power, we discover that the noise power bias of a “complex, magnitude-squared” averaging device is:

\[
\text{BIAS}_{\text{complex magnitude-squared}} \text{ [dB]} = 10\log_{10} \left( \frac{P_{\text{complex magnitude-squared}}}{P_0} \right) = 10\log_{10} \left( \frac{\sigma^2}{\sigma^2} \right) = 0 \text{ dB}.
\]  

(39)

Therefore, if averaging is performed using the magnitude-squared of the complex signal, the noise power can be computed without bias.

The standard deviation of a single sample of a “complex, magnitude-squared” device can be computed knowing that \(|X|^2\) is a chi-squared distribution with two degrees of freedom, where each of the underlying normal distributions have a variance of \( \sigma^2/2 \). The standard deviation can be computed as follows [20]:

\[
\sigma_{\text{complex, magnitude-squared}} = \sqrt{\text{Var}[|X|^2]} = \sqrt{\text{Var}[\frac{\sigma^2}{2} \chi^2(2)]} = \sqrt{\left( \frac{\sigma^2}{2} \right) \cdot 2 \cdot 2} = \sigma.
\]  

(40)

Thus, the standard deviation of a reported value after averaging \( N \) samples using a “complex, magnitude-squared” device is given as:

\[
\sigma_{\text{complex, magnitude-squared}} \left( N \text{ samples} \right) = \frac{\sigma}{\sqrt{N}}.
\]  

(41)

This method achieves the Cramer-Rao Lower Bound (CRLB) for a standard deviation estimator of complex white Gaussian noise [13], [22].

Using (13), the uncertainty in dB is given by:

\[
U_{\text{complex, magnitude-squared}} = 10\log_{10} \left( \frac{1 + k}{\sqrt{N}} \right)
\]  

(42)

\[
= 10\log_{10} \left( \frac{1 + k}{\sqrt{N}} \right) \approx \pm 10k \frac{1}{\sqrt{N}} \log_{10} e \approx \pm \frac{4.3429k}{\sqrt{N}} \text{ dB}.
\]

Magnitude Averaging of Complex Noise

If the noise is sampled with a device similar to a swept-tuned superheterodyne spectrum analyzer with the signal routed through the linear path (Fig. 3), then the output of the envelope detector is \( |X| \). If this device were to report power by first taking the time-average of the output, and then squaring that result, this is an example of a device that performs “complex, magnitude” averaging.

To determine the bias incurred by employing “complex, magnitude” averaging, we must first consider the distribution of \( |X| \). The magnitude of the signal \( X \) is given as:

\[
|X| = \sqrt{I^2 + Q^2}.
\]  

(43)

Thus, the distribution of the magnitude of \( X \) is a Rayleigh distribution, otherwise known as a chi distribution with two degrees of freedom, where each underlying normal distribution has a variance of \( \sigma^2/2 \), see (4) and (5). The mean of this distribution is given as [20]

\[
E[|X|^2] = \frac{\sigma^2}{2} \cdot \frac{\pi}{2} = \frac{\sigma^2 \pi}{4}.
\]  

(44)

Thus, the noise power value reported by a “complex, magnitude” averaging device is given as follows:

\[
P_{\text{complex magnitude}} = E[|X|^2] = \frac{\sigma^2 \pi}{4}.
\]  

(45)

Comparing this result to the true noise power, we discover that the noise power bias of a “complex, magnitude” averaging device in dB is:

\[
\text{BIAS}_{\text{complex magnitude}} \text{ [dB]} = 10\log_{10} \left( \frac{P_{\text{complex magnitude}}}{P_0} \right) = 10\log_{10} \left( \frac{\sigma^2}{\sigma^2} \right) \approx -1.0491 \text{ dB}.
\]  

(46)

Thus, a “complex, magnitude” averaging device may be used to estimate noise power as long as 1.0491 dB is added to the reported result. This is in agreement with the results shown in [2] and [16].
The standard deviation of a single measurement reported
by a “complex, magnitude” device can be computed by know-
ing that the distribution of $|X|$ is a Rayleigh distribution where
each underlying normal distribution has a variance of $\sigma_z^2 / 2$
[20]:

$$\sigma_{\text{complex,magnitude}} = \sqrt{\text{Var}[|X|]} = \sqrt{\frac{4 - \pi \sigma_z^2}{2}} = \sigma_z \sqrt{\frac{1 - \pi}{4}}. \quad (47)$$

Therefore, the standard deviation of the reported value
when averaging $N$ samples using a “complex, magnitude”
device is given as:

$$\sigma_{\text{complex,magnitude}}(N \text{ samples}) = \sigma_z \sqrt{\frac{1 - \pi}{4 \sqrt{N}}}. \quad (48)$$

From (13), the k-$\sigma$ uncertainty is given by:

$$U_{\text{complex,magnitude}} = 20 \log_{10} \left[ 1 \pm k \frac{\sigma_z \sqrt{1 - \pi}}{\sqrt{N}} \right]. \quad (49)$$

$$= 20 \log_{10} \left[ 1 \pm \frac{4 - 1}{\sqrt{N}} \right] \approx \pm 20 k \frac{4 - 1}{\sqrt{N}} \log_{10} e \approx \pm 4.5 \frac{50 \pi^2}{\sqrt{N}} \text{dB.}$$

Logarithmic Averaging of Complex Noise

Let us now assume that the measurement device computes
the logarithm of the complex signal prior to performing the aver-
ing operation. This is an example of “complex, logarithmic”
averaging. This would be the case if employing a swept-tuned
superheterodyne spectrum analyzer with the log-amp path
selected, such as the architecture shown in Fig. 3. Also, a real-
time spectrum analyzer, such as the one depicted in Fig. 4, may
choose to implement this type of averaging in either firmware
or software.

To begin our analysis, the random variable $X$ is converted
to dB as:

$$Y_c = 20 \log_{10} (|X|). \quad (50)$$

The variable $Z_c = |X|$ is Rayleigh-distributed, so it con-
forms to the following probability density function:

$$f_{Z_c}(z) = \frac{2z}{\sigma_{Z_c}^2} e^{-z^2/\sigma_{Z_c}^2} \quad (51)$$

where each underlying normal distribution has a variance of $\sigma_z^2 / 2$.

Using this knowledge of the distribution of $|X|$, the expected value of $Y_c$ is derived in Appendix C and is given by:

$$E[Y_c] = E\left[20 \log_{10} (|X|)\right] = \int_{0}^{\infty} (20 \log_{10} (z)) \frac{2z}{\sigma_x^2} e^{-z^2/\sigma_x^2} dz$$

$$= 10 \log_{10} \left(\frac{\pi}{\ln(10)}\right) - 0.5 \sigma_x^2 = 2.5068 \text{dB}. \quad (52)$$

Comparing this result to the true noise power, we discover
that the noise power bias of a “complex, logarithmic” averag-
ing device is:

$$\text{BIAS}_{\text{logarithmic}} \left[\text{dB}\right] = E[20 \log_{10}(|X|)] - 10 \log_{10}(E[\sigma_x^2])$$

$$= 10 \log_{10}(\sigma_x^2) - \frac{10}{\ln(10)} \gamma = -2.5068 \text{dB.} \quad (53)$$

Thus, a “complex, logarithmic” averaging device may be
used to estimate noise power as long as 2.5068 dB is added to
the reported result. This is in agreement with the result shown
in [2].

To estimate the uncertainty of the value reported by a “com-
plex, logarithmic” device, we again must first compute the
standard deviation of a single instance of $Y_c$. The standard
device is computed by first finding the second moment. This
is derived in Appendix D and is given by:

$$E[Y_c^2] = E\left[20 \log_{10} (|X|)^2\right] = \int_{0}^{\infty} \left(20 \log_{10}(z)^2\right) \frac{2z^2}{\sigma_x^2} e^{-z^2/\sigma_x^2} dz$$

$$= \frac{400}{\ln(10)} \left[ \ln^2(\sigma_x) - \gamma \ln(\sigma_x) + \frac{\pi^2}{4} \right] \quad (54)$$

The expected value of $Y_c$ was computed previously. There-
fore, by employing (52) and (54), we find that:

$$\sigma_{Y_c}^2 = E[Y_c^2] - E[Y_c]^2 = \frac{50}{3} \frac{\pi^2}{\ln^2(10)} \quad (55)$$

The standard deviation is therefore:

$$\sigma_{Y_c} = \sqrt{\frac{50}{3} \frac{\pi}{\ln(10)}} = 5.57 \text{dB.} \quad (56)$$

As with a “real, logarithmic” device, the variance and stand-
ard deviation of measured noise power from a “complex, logarithmic” device are independent of the noise power level.
As before, the k-$\sigma$ uncertainty of an average of $N$ samples is:

$$U_{\text{complex,logarithmic}} = \sqrt{\frac{50}{3} \frac{\pi}{\ln(10)}} \frac{k}{\sqrt{N}} = \frac{5.5700k}{\sqrt{N}} \text{dB.} \quad (57)$$

Summary of the Biases and Uncertainties of Noise Power Measurements

The biases for the six analyzed cases are summarized in Ta-
ble 2.

To verify the analytic expressions, a Monte Carlo simula-
tion was conducted with 107 realizations of real and complex
noise processes. Table 3 summarizes the results. Deviations from the analytic expressions are given in parentheses.

The uncertainty expressions are approximately of the same form for all six cases. For four of the cases, we truncated a Taylor series to simplify the expression, which is valid as the number of samples we average becomes large. In practice, using $N \geq 100$ will yield very good approximations with these expressions.

All of the simplified expressions follow the form:

$$U = \pm \frac{A_k}{\sqrt{N}} \text{dB}.$$  \hspace{1cm} (58)

The value of $A$ varies for each of the six cases. Table 4 shows the values of $A$ for each of the six cases.

A Monte Carlo simulation was run with $N = 10^4$ averages and $10^6$ realizations of noise processes. Table 5 summarizes the simulation’s ensemble of standard deviation values of the sample mean. The values in the table are normalized such that they correspond to the values of $A$ given in (58). The actual uncertainties for a 1-σ deviation are smaller by a factor of $\sqrt{N} = 100$. Deviations from the analytic expressions are given in parentheses.

Notice that the uncertainties are smaller for complex noise than for real noise. The magnitude-squared and magnitude averaging have lower uncertainties than logarithmic averaging. The magnitude-squared method, which is the only unbiased method, has the lowest uncertainty of the three, and is probably the best choice if available on the instrument being used to make the noise power measurement.

Fig. 5 summarizes the 1-σ uncertainty values for the six different methods for different values of $N$.

**Conclusions**

Various RF instruments are available for making noise power measurements, but the system architecture and processing methods of these instruments may yield different results. In this paper, we explored some of the RF instruments that are commonly used for such measurements and analyzed their biases and uncertainties. With knowledge of the bias, the user can correct for it. With knowledge of the uncertainty, the user can make an informed decision about the number of averages that are needed to achieve the desired measurement uncertainty. If given a choice, using the “complex, magnitude-squared” averaging method is recommended as it will result in an unbiased result with the smallest measurement uncertainty.
In all cases, both the bias and uncertainty, given in dB, are independent of the true noise power being measured.

Acknowledgment
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References
[15] “Fundamentals of RF and microwave power measurements,” Agilent Application Note 64-1C.
In this appendix, we calculate the mean of the logarithm of a real noise signal to be:

$$E[Y_{x}] = \int_{-\infty}^{\infty} 20 \log_{10}[|h|] e^{\frac{-x^2}{2\sigma_x^2}} dx$$

(59)

$$= 20 \log_{10}[\sigma_x] - 10 \ln(10) \left[ \gamma + \ln(2) \right]$$

(60)

We begin by re-arranging the integral to use natural logarithms and take advantage of the Gaussian distribution’s symmetry about zero.

$$E[Y_{x}] = \int_{-\infty}^{\infty} 20 \log_{10}[|h|] e^{\frac{-x^2}{2\sigma_x^2}} dx$$

(61)

Using the u-substitution of $u = \frac{x}{\sqrt{2\pi} \sigma_x}$ to evaluate the integral, we find:

$$\int_{-\infty}^{\infty} 20 \log_{10}[|h|] e^{\frac{-x^2}{2\sigma_x^2}} dx = \int_{-\infty}^{\infty} 20 \log_{10}[\sigma_x] e^{-u^2} du$$

(62)

The half integral of a zero-mean, Gaussian distribution with variance of 1/2 is given by:

$$\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = \frac{1}{2}$$

(63)

Therefore, we can evaluate the first integral of (61) as follows:

$$\int_{0}^{\infty} 20 \log_{10}[\sigma_x] e^{-u^2} du = -\frac{\sqrt{\pi}}{2} \left[ \gamma + \ln(4) \right]$$

(64)

By inserting the results of (65) into (61), we find that:

$$E[Y_{x}] = 20 \log_{10}[\sigma_x] - 10 \ln(10) \left[ \gamma + \ln(2) \right]$$

(66)

Equation 4.333 of [23] offers the solution to the second integral of (61):

$$\int_{0}^{\infty} \ln(u) e^{-u^2} du = -\frac{\sqrt{\pi}}{2} \left[ \gamma + \ln(4) \right]$$

(67)

where $\gamma \approx 0.5772157$ is the Euler-Mascheroni constant.

Inserting (63) and (64) into (61), we find that:

$$E[Y_{x}] = 20 \log_{10}[\sigma_x] - 10 \ln(10) \left[ \gamma + \ln(2) \right]$$

(68)

As in Appendix A, we use the u-substitution of $u = \frac{x}{\sqrt{2\pi} \sigma_x}$ to reformulate the integral:

$$\int_{-\infty}^{\infty} 20 \log_{10}[|h|] e^{\frac{-x^2}{2\sigma_x^2}} dx = \int_{-\infty}^{\infty} 20 \log_{10}[\sigma_x] e^{-u^2} du$$

(69)

$$= \int_{-\infty}^{\infty} 20 \log_{10}[\sigma_x] e^{-u^2} du$$

(70)

$$= \frac{800}{\sqrt{2\pi} \sigma_x \ln(10)} \int_{0}^{\infty} \ln^2(u) e^{-u^2} du$$

(71)

In this appendix, we calculate the second moment of the logarithm of real noise to be:

$$E[Y_{x}^2] = \int_{-\infty}^{\infty} 20 \log_{10}[|h|]^2 e^{\frac{-x^2}{2\sigma_x^2}} dx$$

(72)

$$= \frac{400}{\sqrt{2\pi} \sigma_x \ln(10)} \int_{0}^{\infty} \ln^2(u) e^{-u^2} du$$

(73)

$$= \frac{400}{\sqrt{2\pi} \sigma_x \ln(10)} \left[ \gamma + \ln(4) \right]$$

(74)

We begin by re-arranging the integral to use natural logarithms and take advantage of the Gaussian distribution’s symmetry about zero.

$$E[Y_{x}^2] = \int_{-\infty}^{\infty} 20 \log_{10}[|h|]^2 e^{\frac{-x^2}{2\sigma_x^2}} dx$$

(75)

$$= \frac{800}{\sqrt{2\pi} \sigma_x \ln(10)} \int_{0}^{\infty} \ln^2(u) e^{-u^2} du$$

(76)

$$= \frac{800}{\sqrt{2\pi} \sigma_x \ln(10)} \int_{0}^{\infty} \ln^2(u) e^{-u^2} du$$

(77)
Solutions for the first two integrals of (69) are given by (63) and (64). For the third integral, we use equation 4.335 of [23] to yield:

$$\int_0^\infty \ln^2(u) e^{-du} = \frac{\sqrt{\pi}}{8} \left( \gamma + 2\ln 2 + \frac{\pi^2}{2} \right)$$

Inserting (63), (64), and (70) into (69), we find that:

$$\int_0^\infty \ln^2(u) e^{-du} = \frac{\sqrt{\pi}}{8} \left( \gamma + 2\ln 2 + \frac{\pi^2}{2} \right)$$

Finally, by inserting (71) into (68), we find the second moment of the logarithm of real noise to be:

$$E[Y_r^2] = \frac{800}{\sqrt{2\pi} \sigma_x \ln(10)} \left\{ \ln^2(\sqrt{2\pi} \sigma_x) - \ln(\sqrt{2\pi} \sigma_x) \right\} \left( \gamma + 2\ln 2 + \frac{\pi^2}{8} \right)$$

$$= \frac{400}{\ln(10)} \left\{ \ln^2(\sqrt{2\pi} \sigma_x) - \ln(\sqrt{2\pi} \sigma_x) \right\} \left( \gamma + 2\ln 2 + \frac{\pi^2}{8} \right)$$

$$= \frac{400}{\ln(10)} \left\{ \ln^2(\sqrt{2\pi} \sigma_x) - \ln(\sqrt{2\pi} \sigma_x) \right\} \left( \gamma + 2\ln 2 + \frac{\pi^2}{8} \right)$$

Finally, inserting (78) into (74), we find the mean of the logarithm of a complex Gaussian to be:

$$E[Y_c] = \frac{400}{\ln^2(10)} \left\{ \ln^2(\sqrt{2\pi} \sigma_x) - \ln(\sqrt{2\pi} \sigma_x) \right\} \left( \gamma + 2\ln 2 + \frac{\pi^2}{24} \right)$$

To evaluate the integral, we use the u-substitution of \( u = \frac{x^2}{\sigma_x^2} \) as follows:

$$\int_0^\infty \ln^2(u) e^{-du} = \frac{\sqrt{\pi}}{8} \left( \gamma + 2\ln 2 + \frac{\pi^2}{2} \right)$$

The first integral shown in (75) is trivial and evaluates to 1:

$$\int_0^\infty e^{-du} = 1$$

Finally, by inserting (71) and (77) into (75), we find that:

$$\ln(u) e^{-du} = -\gamma$$

where \( \gamma = 0.5772157 \) is the Euler-Mascheroni constant.

Finally, inserting (78) into (74), we find the mean of the logarithm of a complex Gaussian to be:

$$E[Y_c] = \frac{400}{\ln^2(10)} \left\{ \ln^2(\sqrt{2\pi} \sigma_x) - \ln(\sqrt{2\pi} \sigma_x) \right\} \left( \gamma + 2\ln 2 + \frac{\pi^2}{24} \right)$$

To begin, we write the above integral using natural logarithms:

$$E[Y_c^2] = \int_0^\infty \ln^2(u) e^{-du} = \frac{800}{\ln^2(10)} \left\{ \ln^2(\sqrt{2\pi} \sigma_x) - \ln(\sqrt{2\pi} \sigma_x) \right\} \left( \gamma + 2\ln 2 + \frac{\pi^2}{24} \right)$$

To evaluate the integral of (81), we use the u-substitution of \( u = \frac{x^2}{\sigma_x^2} \) as follows:
\[
\int_0^\infty \ln^2(z) e^{-\frac{z^2}{2}} dz = \frac{\sigma_x^2}{2} \int_0^\infty \left( \ln^2(\sqrt{2\pi} \sigma_x) \right) e^{-u} du
\]

\[
= \frac{\sigma_x^2}{2} \left[ \int_0^\infty \ln^2(\sigma_x) + \ln(\sigma_x) \ln(u) + \frac{1}{4} \ln^2(u) \right] e^{-u} du
\]

\[
= \frac{\sigma_x^2}{2} \left[ -\frac{1}{2} \ln^2(\sigma_x) + \ln(\sigma_x) \ln(u) + \frac{1}{4} \ln^2(u) \right] e^{-u} du + \frac{1}{4} \sigma_x^2 \int_0^\infty \ln^2(u) e^{-u} du
\]

(82)

The first two integrals were evaluated in (76) and (77), found in Appendix C.

Using equation 4.3.35 (91) from [23] to evaluate the third integral, we find that:

\[
\int_0^\infty \ln^2(u) e^{-u} du = \left( \frac{\pi^2}{6} + \gamma^2 \right)
\]

(83)

Inserting (76), (77), and (83) into (82), we find that:

\[
\int_0^\infty \ln^2(z) e^{-\frac{z^2}{2}} dz
\]

\[
= \frac{\sigma_x^2}{2} \left[ \ln^2(\sigma_x) \left( 1 + \ln(\sigma_x) \right) - \frac{\gamma^2}{4} \right] + \frac{\pi^2}{6} + \gamma^2
\]

(84)

Finally, inserting (84) into (81), we find the second moment of the logarithm of a complex Gaussian to be:

\[
E[Y^2] = -\frac{800}{\ln^2(10) \sigma_x^4} \left[ \ln^2(\sigma_x) - \gamma \ln(\sigma_x) + \frac{\gamma^2}{4} \right] + \frac{\pi^2}{24}
\]

(85)