A NOTE TO SHOW HOW AN ALTERNATIVE SPHERICAL MODE NORMALIZATION SIMPLIFIES THE RELATIONSHIP BETWEEN TRANSMITTING AND RECEIVING CHARACTERISTICS

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ABSTRACT

In this paper we show how a modification in the choice of mode normalization changes the pair-wise transmitting-receiving conversion to a one-to-one equality for a reciprocal antenna:

$$\overline{R}^{Rcv}_{smn} = \overline{T}^{Xmt}_{smn}$$

This change affords us greater simplicity and the opportunity to avoid confusion when manipulating the scattering coefficients. For this relation to hold, there is a useful convention defining two fiducial coordinate systems for the antenna – one a transmitting and the other a receiving coordinate system.

Keywords: Spherical Modes, Spherical Near-Field Scanning, Reciprocity, Fiducial Coordinates, Near-Field Probes.

1. Introduction

The problem of computing the transfer of excitation between two antennas is a problem that underlies all of antenna measurements [1,2,3,4]. This two-antenna coupling problem is set up by defining fiducial coordinates for each antenna and specifying coordinate parameters that describe the length R of the line that connects the two origins and the orientation directions θ_r, ϕ_r and θ_t, ϕ_t for the receiving and transmitting antennas respectively. We are very used to seeing the case of far-field coupling, described by the Friis transmission equation for the impedance-matched, polarization-matched conditions, written as [1]

$$\frac{P_r}{P_t} = (\frac{\lambda}{4\pi R})^2 G_t(\theta_t, \phi_t) G_r(\theta_r, \phi_r) . \qquad (1)$$

This, even though the term *gain* is strictly applicable only for a transmitting antenna. We have become accustomed to the tacit assumption that the conversion factor $(4\pi/\lambda^2)$ will always have been used to convert the *effective area* parameter, A_{eff}, for a reciprocal antenna into the corresponding transmit parameter, *gain*.

We almost always write this far-field transmission equation in terms of two transmit parameters the gain of each antenna. Contrast this with the current practice in stating the spherical near-field (SNF) transmission equation [3,4]:

$$\frac{w'(r_0;\varphi_0,\theta_0,\chi_0)}{v} = \sum_{\substack{s\mu mn \\ \sigma\nu}} R'_{\sigma\mu\nu} \mathcal{C}^{sn}_{\sigma\mu\nu}(kr_0\hat{z}) \mathcal{D}^{(n)}_{\mu m}(\varphi_0,\theta_0,\chi_0) T_{smn}$$
(2)

This equation expresses the ratio of output voltage at the receiving antenna to the input voltage at the transmitting antenna as a function of the range coordinate geometry $r_0 ; \theta_0, \phi_0, \chi_0$, the wave-number k, and the characteristics of the respective antennas. The parameter set $R'_{\sigma\mu\nu}$ that characterize the receiving probe antenna and the parameter set T_{smn} that characterizes the transmitting test antenna refer specifically to the transmit/receive state of the respective antennas. Depending on the convention(s) adopted in deriving this equation, there is always a known conversion factor for reciprocal antennas that can be used to relate the two types of characteristic. (More about this follows below.) For reciprocal antennas, two sets of equivalent quantities is redundant.

In this paper we show that by choosing an alternative normalization factor for the vector spherical modes, the same characteristic can be used equivalently as both a transmit characteristic and a receive characteristic [4]. This change affords us the simple expression

$$\overline{R}^{Rcv}_{smn} = \overline{T}^{Xmt}_{smn} , \qquad (3)$$

and, as a result, the opportunity to avoid confusion when manipulating the scattering coefficients. (The over-bar indicates this alternative normalization.) It relieves us of the necessity of carrying a redundant description of reciprocal antennas. Just as for the gain parameter currently, one characteristic alone is sufficient to describe a reciprocal antenna. For relation (3) to hold, associated with this alternative normalization, is a useful convention to define the antenna's fiducial coordinate systems, which we also discuss.

2. Background

The family of spherical modes that underlie the transmission equation on which spherical near-field scanning is based must satisfy multiple requirements for the theory to be workable. Each mode must satisfy at least the following stipulations:

i. Represents a Vector Solution to the Free-Space Maxwell Equations
ii. Possesses a Convenient Power Normalization
iii. Obeys the Rotation Theorem
iv. Obeys the Translation Theorem
v. Obeys a Known Reciprocity Rule

A standard form of the spherical modes is documented in the reference "Spherical Near-Field Antenna Measurements" edited by J.E. Hansen. [5] We review for the reader Hansen's definition of the spherical modes in Appendix B. Please see equations (B1), (B2), (B3).

Hansen employs the concept of a scattering matrix to represent the general characteristic of an arbitrary linear antenna. The corresponding scattering matrix transmitting and receiving characteristics are written as T_{smn} and R_{smn}, respectively [6]. Again, the subscripts *s,m,n* are the spherical modal indices that serve as labels for the various vector spherical modes.

The scattering matrix assumption divides the modal set into two pieces – one piece consists of a family inward traveling waves and the other a family of outward traveling waves. This separation permits a scattering matrix equation to be written that relates the outward going wave amplitudes w, b_{smn} to the inward going wave amplitudes v, $a_{s'm'n'}$. Here v and w refer to port excitation voltages.

$$\begin{bmatrix} \Gamma_{0,0} & R_{0,s'm'n'} \\ T_{smn,0} & S_{smn,s'm'n'} \end{bmatrix} \bullet \begin{bmatrix} v \\ a_{s'm'n'} \end{bmatrix} = \begin{bmatrix} w \\ b_{s'm'n'} \end{bmatrix} \qquad (4)$$

Hansen shows by making use of the Lorentz reciprocity theorem for a reciprocal antenna [7], that

$$R_{smn} = (-)^m T_{s,-m,n} \ , \qquad (5)$$

i.e. the transmitting and receiving characteristics are pair-wise related by a factor $(-1)^m$. The spherical modes defined in (4) and (5) above satisfy all of the conditions (*i*) – (*v*). Equation (5) exhibits the rule for condition (*v*).

3. Hansen's Choice of Normalization

Hansen's exposition of spherical near-field to far-field transformation theory utilizes a spherical mode normalization that gives the power in a transmitted mode as ½ times the square of the modulus of the complex modal coefficient corresponding to that mode [8]. This normalization factor is chosen to make the expression for total transmitted power as simple as possible.

$$P = \tfrac{1}{2} \sum_{smn} \left| Q_{smn} \right|^2 \ , \qquad (6)$$

where Q_{smn} is the complex modal amplitude for mode *s,m,n* in the expansion of the antenna's electric field. With the alternative normalization proposed below, equation (6) continues to hold; for this to be true, the renormalization factor must have a modulus of unity !

4. The Alternative Choice of Normalization

If we choose to take as the normalization constant for the scalar generating function the same constant exhibited in (B2) except with the addition of a simple factor i^n, we will have a new generating function with an alternative normalization

$$\overline{F}^{(c)}_{nm}(r,\theta,\phi) = \frac{i^n}{\sqrt{2\pi\, n(n+1)}} \left(-\frac{m}{|m|}\right)^m z_n^{(c)}(kr) \overline{P}_n^{|m|}(\cos\theta) e^{im\phi} \qquad (7)$$

We denote this new scalar function on the left-hand side with an over-bar. The renormalized vector spherical modes will correspondingly carry this factor. (This choice of normalization has a precedent in quantum physics [9,10]; also, Wittmann has used it in his exposition of SNF theory [11], [12].)

As we show in Appendix B, this leads to a new relationship between the transmitting and receiving characteristics of a reciprocal antenna expressed in a single coordinate system:

$$\overline{R}_{smn} = (-)^{n-m} \overline{T}_{s,-m,n} \qquad (8)$$

We next examine the case of two coordinate systems.

5. Coordinates Systems for Transmitting and Receiving Antennas

There are in practice *two* different fiducial coordinate systems used to describe probe antennas in near-field scanning. In Appendix A, we show for planar near-field scanning how these can be thought of as transmitting and receiving coordinates. In spherical near-field scanning also, the issue appears when dealing with the probe antenna [13]. This same observation can be made with respect to antennas in general: There are two coordinate systems used to describe the fields surrounding an antenna – one when it is transmitting and one when it is receiving. Two typical schematics with different antenna coordinates that illustrate this are shown below.

We make here the following observation: It is convenient and appropriate to have the following convention regarding coordinate systems: "When describing the transmitting characteristics of an antenna, the fiducial *transmitting* coordinate system shall have the z-axis exiting the aperture and when describing the receiving characteristic of that antenna the fiducial *receiving* coordinate system shall have the z-axis entering the aperture."

The relationship between the transmitting and receiving coordinate axes is a 180° rotation about the y-axis as can be understood from examination of Figures 1 and 2. We now investigate the implication of this fact.

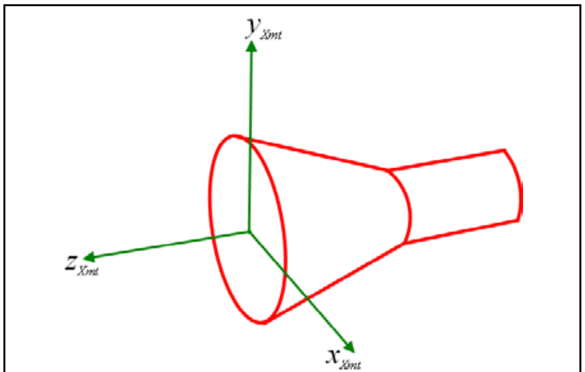

Figure 1. Transmitting Coordinate System

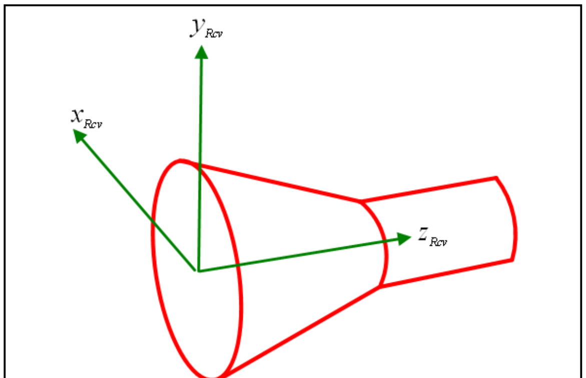

Figure 2. Receiving Coordinate System

6. Coordinates and Transmitting and Receiving Characteristics

Consider a general antenna described using the alternative mode normalization in terms of its transmitting and receiving characteristics, but described in its natural or transmitting coordinate system with the z-axis exiting the aperture – i.e. the transmitting coordinate system.

The transmitting characteristic can also be expressed in the receiving coordinate system that is obtained by a 180° rotation about the y-axis. The equation for this conversion is a simple matrix equation [14,15]:

$$\overline{T}^{Rot}_{sm'n} = \sum_m d^{(n)}_{m'm}(\pi)\overline{T}_{smn} \ , \qquad (9a)$$

where $$d^{(n)}_{m'm}(\pi) = (-)^{n+m}\delta_{m',-m} \ ; \qquad (9b)$$

and where $\overline{T}_{smn}$ is the transmitting characteristic in the transmitting coordinate frame and $\overline{T}^{Rot}_{smn}$ is the transmitting characteristic in the rotated – i.e. receiving – coordinate frame. Thus

$$\overline{T}^{Rot}_{sm'n} = $$
$$= \sum_m d^{(n)}_{m'm}(\pi)\overline{T}_{smn} = \sum_m (-)^{n+m}\delta_{m',-m}\overline{T}_{smn} = (-)^{n-m'}\overline{T}_{s-m'n} \qquad (10)$$

or $$\overline{T}^{Rot}_{smn} = (-)^{n-m}\overline{T}_{s,-m,n} \qquad (11)$$

Using equation (10) above, we can also write an expression for the receiving coefficients in the rotated coordinates.

$$\overline{R}^{Rot}_{smn} = (-)^{n-m}\overline{T}^{Rot}_{s,-m,n} \ . \qquad (12)$$

Now we can relate the rotated transmitting characteristic to the original transmitting characteristic by combining equations (11) and (12):

$$\overline{R}^{Rot}_{smn} = (-)^{n-m}\overline{T}^{Rot}_{s,-m,n} = (-)^{n-m}[(-)^{n+m}\overline{T}_{s,m,n}] = \overline{T}_{s,m,n} \qquad (13)$$

That is, for a reciprocal antenna, the receiving coefficients in the rotated frame are equal to the transmitting coefficients in the original frame when this alternative mode normalization is employed. Or, said another way, the transmitting coefficients in the transmitting coordinate system are equal to the receiving characteristics in the receiving coordinate system! This is a very useful result because it also clarifies the relationship between transmitting and receiving coordinates and which coordinate system is used to express which characteristic. We employ the convention that the transmitting and receiving coordinates are as illustrated in Figures 1 & 2, for the transmitting and receiving states of the antenna.

$$\overline{R}^{Rcv}_{smn} = \overline{T}^{Xmt}_{smn} \ . \qquad (14)$$

7. Summary - Advantages of Alternative Normalization and Coordinates

The advantage of the normalization convention proposed here is the economy realized for reciprocal antennas and the convenience of consistent coordinates. This convention leads to a simplification in manipulating antenna characteristics for reciprocal antennas. Redundant data can be eliminated. This choice of alternative normalization requires no change in the form of the SNF transmission equation – only a minor change in the modal translation coefficients of the translation theorem. We state without proof that this change in fact simplifies the translation theorem by eliminating two of the i^n factors. The convention of the transmitting and receiving coordinates guarantees that both members of a coupled pair are represented with parallel axes, avoiding the need for additional rotation.

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Appendix A
Antenna Coordinate Systems for Planar Near-Field Scanning

Please refer to Figure A-1 below.

In setting up the transmission equation for planar near-field scanning, Kerns [2] establishes a single coordinate system and places the test antenna and the probe antenna on two locations along the z-axis. The test antenna is placed at z=0 and the probe at z = +d. The free space region between the test antenna and the probe contains the modes that propagate between the pair of antennas – from the transmitting test antenna toward the receiving probe antenna. The far field of the test antenna lies behind and beyond the probe antenna

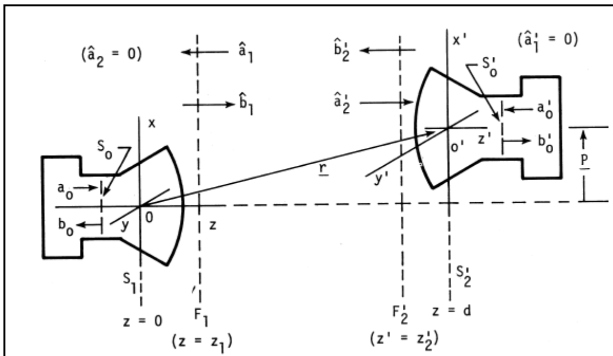

Figure A-1. Schematic of Coupled Two-Antenna System, Typical of Antenna Under Test Plus a Probe

where z→∞.

The usual x-,y-,z- coordinate system can be described as the *transmitting* coordinate system of the antenna-under-test; it is in the related spherical coordinate analog of this x-,y-,z- coordinate system that the antenna's far field is expressed.

It is common practice to express the fields of a transmitting antenna in a coordinate system whose z-axis is directed out of the antenna aperture. Here we call this the *transmitting* coordinate system. Please see Figure A-2 for an illustration of the transmitting coordinate system for a probe. This is the natural definition when measuring such a probe antenna on a

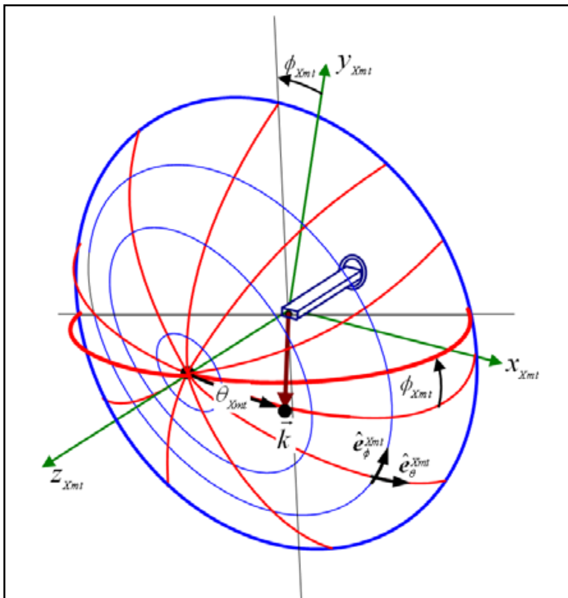

Figure A-2. Schematic of Transmitting Coordinate System Illustrated for a Probe Antenna

roll-over-azimuth test positioner.

It has been found that in the practice of making PNF measurements, one needs to define another coordinate system for the probe antenna when it is receiving [16]. This other coordinate system is termed here the *receiving* coordinate system of the probe. It is also called the *probe's coordinate system*. The need is brought about by the necessity of expressing the probe's receiving characteristic in the coordinate system of the antenna under test. This is illustrated in the schematic of Figure A-1. Please see Figure A-3 for the probe's receiving coordinate system.

We now know from experience that defining both a transmitting and a receiving coordinate system is a practical necessity for the case of a PNF probe antenna.

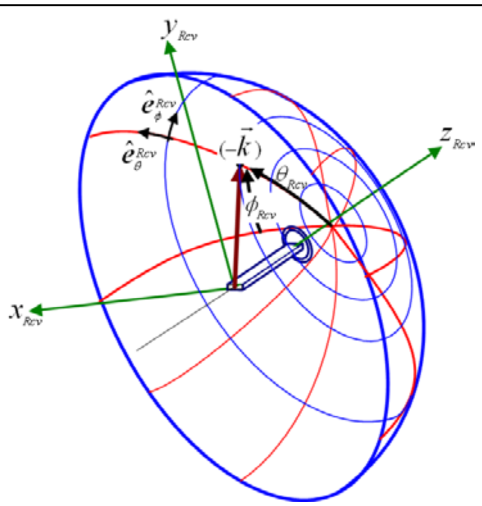

Figure A-3. Schematic of Receiving Coordinate System Illustrated for a Probe Antenna

Appendix B
The Relationship Between Transmitting and Receiving Characteristics in Spherical NF Theory

Vector Spherical Modes of Hansen

To expand the free space solutions to the Maxwell equations, Hansen employs pairs of vector modes in spherical coordinates defined below [5]. Each mode is labeled by three indices – *s, m, n*. The index s = 1 & 2. The value s=1 corresponds to a purely transverse vector field whereas s=2 to one that possesses a radial component. i.e. s=1 connotes a TE-wave and s=2 a TM wave in the expression for electric field. The fields are written

$$\vec{E}(r,\theta,\phi) = \frac{k}{\sqrt{\eta}} \sum_{c,s,m,n} Q^{(c)}_{smn} \vec{F}^{(c)}_{smn}(r,\theta,\phi)$$

$$\vec{H}(r,\theta,\phi) = -ik\sqrt{\eta} \sum_{c,s,m,n} Q^{(c)}_{smn} \vec{F}^{(c)}_{3-s,mn}(r,\theta,\phi) \quad \text{(B1)}$$

where $k=\omega\sqrt{\mu\varepsilon}$ is the wave number and $\eta=\sqrt{\varepsilon/\mu}$ the admittance of the medium. The index *n* indicates the degree of the wave function and the index *m* the order of the wave function. In Hansen's notation, the index *c* takes on one of four values – *1,2,3,4*. Importantly, c = 3,4 corresponds to the radial dependence of $\vec{F}^{(c)}_{smn}(r,\theta,\phi)$ being a spherical Hankel function of the first or second kind, respectively.

The vector wave functions are derived from a scalar generating function that is a solution to the scalar free-space Helmholtz equation. In Hansen's notation, the standard scalar function is written as

$$F^{(c)}_{mn}(r,\theta,\phi) = \frac{1}{\sqrt{2\pi\, n(n+1)}} \left(-\frac{m}{|m|}\right)^m z^{(c)}_n(kr)\overline{P}^{|m|}_n(\cos\theta)e^{im\phi} \quad \text{(B2)}$$

The modal indices in (B1) & (B2), *m & ,n*; are such that $1 \le n \le n_{max} \equiv (\pi D_{min})/\lambda$, and $-n_{max} \le m \le +n_{max}$. The special functions are $z_n^{(c)}(kr)$, a spherical Bessel function, $\overline{P}^m_n(\cos\theta)$ a normalized Legendre polynomial function, and $e^{im\phi}$, the usual sinusoidal exponential function. Here the wave number $k \equiv (2\pi/\lambda)$, and r,θ,ϕ are the usual spherical coordinates. D_{min} is the diameter of the minimum sphere, and λ the free-space wavelength.

The vector wave functions for s=1 & 2 are written, following Hansen [5], as

$$\vec{F}^{(c)}_{1mn}(r,\theta,\phi) = \nabla F^{(c)}_{mn}(r,\theta,\phi)\times\vec{r}$$
$$\vec{F}^{(c)}_{2mn}(r,\theta,\phi) = k^{-1}\nabla\times\vec{F}^{(c)}_{1mn}(r,\theta,\phi) \;. \quad \text{(B3)}$$

Vector Spherical Modes with Alternative Normalization

To accomplish the alternative normalization we desire, we re-define the scalar generating function by inclusion of the additional factor i^n. This results in this same additional factor appearing in front of the expressions for the vector modes as well. To distinguish the modes with this alternative normalization, we denote all quantities with an additional over-bar. Thus we write

$$\overline{F}^{(c)}_{mn}(r,\theta,\phi) = \frac{i^n}{\sqrt{2\pi\, n(n+1)}} \left(-\frac{m}{|m|}\right)^m z^{(c)}_n(kr)\overline{P}^{|m|}_n(\cos\theta)e^{im\phi} \quad \text{(B4)}$$

and the same relations as in (3) above hold to yield the vector modal quantities.

$$\overline{F}^{(c)}_{1mn}(r,\theta,\phi) = \nabla\overline{F}^{(c)}_{mn}(r,\theta,\phi)\times\vec{r}$$
$$\overline{F}^{(c)}_{2mn}(r,\theta,\phi) = k^{-1}\nabla\times\overline{F}^{(c)}_{1mn}(r,\theta,\phi) \quad \text{(B5)}$$

We have the simple relationship between the standard and the re-normalized modes:

$$\overline{\vec{F}}^{(c)}_{1mn}(r,\theta,\phi) = i^n(\vec{F}^{(c)}_{1mn}(r,\theta,\phi)$$
$$\overline{\vec{F}}^{(c)}_{2mn}(r,\theta,\phi) = i^n(\vec{F}^{(c)}_{2mn}(r,\theta,\phi) \quad \text{(B6)}$$

The expressions for the fields are equivalent to (1) above except with modified coefficients, which we also denote with over-bars

$$\vec{E}(r,\theta,\phi)=\frac{k}{\sqrt{\eta}}\sum_{c,s,m,n}\overline{Q}_{smn}^{(c)}\ \overline{\vec{F}}_{smn}^{(c)}(r,\theta,\phi)$$

$$\vec{H}(r,\theta,\phi)=-ik\sqrt{\eta}\sum_{c,s,m,n}\overline{Q}_{smn}^{(c)}\ \overline{\vec{F}}_{3-c,m,n}^{(c)}(r,\theta,\phi) \qquad \text{(B7)}$$

Reciprocity Relationship with Modes Having Alternative Normalization

Here we derive the reciprocity relation between the transmitting characteristic and the receiving characteristic for a reciprocal antenna when the alternative normalization is chosen for the free space spherical modes. This derivation is based upon Hansen's [17], which employs the Lorentz reciprocity theorem and the concept of an adjoint antenna.

In applying this theorem two situations are considered:

(1) A set of sources outside of S produces a field denoted $(\overline{\vec{E}},\overline{\vec{H}})$ in the volume V and on the surface S, whose material media are characterized by constants μ,ε .

(2) With the media in V replaced by media having characteristics, μ^T,ε^T a new set of sources, also outside of S, produces another set of fields, $(\overline{\vec{E}}',\overline{\vec{H}}')$.

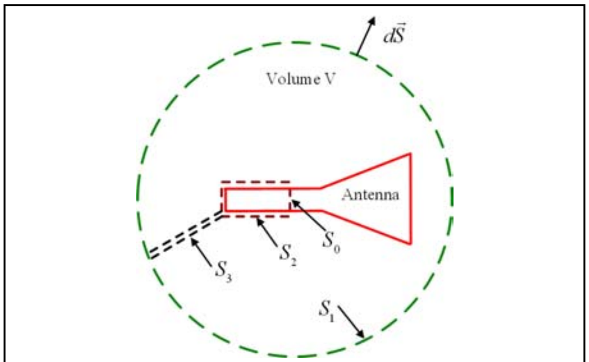

Figure B-1. Surfaces that Comprise the Surface S.

The Lorentz reciprocity theorem states that

$$\int_S(\overline{\vec{E}}\times\overline{\vec{H}}'-\overline{\vec{E}}'\times\overline{\vec{H}})\cdot d\vec{S}=0 \qquad \text{(B8)}$$

The integral over the surface S is broken into four separate integrals. Using the modes with the alternative normalization, each integral gives a result identical to Hansen's except for the expression for S_1 , the integral over the surrounding sphere, which yields

$$\int_{S_1}(\overline{\vec{E}}\times\overline{\vec{H}}'-\overline{\vec{E}}'\times\overline{\vec{H}})\cdot d\vec{S}=-2\sum_{smn}(i^n)^2(-)^m(b_{smn}a_{s,-m,n}'-a_{smn}b_{s,-m,n}')$$

(B9)

The additional factor of i^{2n} comes about from the re-normalization directly. Of course, $i^{2n}=(-)^n$. Hansen shows that the Lorentz form of the reciprocity theorem computed with the spherical modes having standard normalization, leads to the following result:

$$vw'-v'w=\sum_{smn}(-)^m(b_{smn}a'_{s,-m,n}-a_{smn}b'_{s,-m,n})$$

(B10)

When the modes with modified normalization are used, this expression becomes

$$vw'-v'w=\sum_{smn}(-)^{n+m}(b_{smn}a'_{s,-m,n}-a_{smn}b'_{s,-m,n})\ .$$

(B11)

The two sets of fields are assumed to produce specific excitations.

One assumes, following Hansen, that the unprimed quantities – i.e. the first set of fields – are such that

$$v=1 \quad \text{and} \quad a_{smn}=0\text{ , for all } s,m,n \qquad \text{(B12)}$$

and, that the second set of fields (denoted by primed quantities) correspond to the adjoint antenna receiving only one mode. This one mode is designated by indices σ,μ,ν , and it is such that

$$v'=0\ , \qquad \text{(B13a)}$$

and

$$a'_{s,m,n}=1,\quad for\ \ s,m,n=\sigma,\mu,\nu \qquad \text{(B13b)}$$
$$=0,\quad otherwise\ .$$

That is $a'_{smn}=\delta_{s,\sigma}\delta_{m,\mu}\delta_{n,\nu}$. (B13c)

These assumptions and equation (B11) imply that

$$w'=(-)^{\nu-\mu}b_{\sigma,-\mu,\nu}\ . \qquad \text{(B14)}$$

The scattering matrix relations for the two cases imply that

$$\overline{\mathbf{T}}=\boldsymbol{b} \qquad \text{and} \qquad w'=\overline{\mathbf{R}}'\boldsymbol{a}'$$

or

$$\overline{T}_{smn}=b_{smn} \qquad \text{(B15)}$$

and

$$w'=\sum_{smn}\overline{R}'_{smn}\,a_{smn}'=\overline{R}'_{\sigma,\mu,\nu}\ . \qquad \text{(B16)}$$

Substituting from (B15) and (B16) into (B14) gives

$$\overline{R}'_{\sigma,\mu,\nu}=(-)^{\nu-\mu}\overline{T}_{\sigma,-\mu,\nu} \qquad \text{(B17)}$$

In the case of a reciprocal antenna, the primed and unprimed quantities refer to the same antenna and this becomes

$$\overline{R}_{smn}=(-)^{n-m}\overline{T}_{s,-m,n} \qquad \text{(B18)}$$

This then is the relationship we seek, connecting the transmitting and receiving characteristics for a reciprocal antenna under the alternative normalization. It is a pair-wise relationship analogous to Equation (7) in the main body, applicable when a single fiducial coordinate system exists and the alternative normalization is employed.